Matrix Algebra, Calculus, and Probability: Introduction and Review

Monday July 07 (10:00 a.m. - 1:00 p.m.) and Tuesday - Friday July 08 - 11 (8:15 a.m. - 12:15 p.m.), 2025
In-person (1255 North Quad [tentatively; any change will be announced...]) and
Online (with both synchronous and asynchronous access via Canvas and Zoom)

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Introduction

This Lecture workshop is for participants in ICPSR Summer Program courses and will cover:

- Matrix Algebra: Introductory, Intermediate, and Advanced topics
- Differential Calculus: Single-variable and partial derivatives
- Integral Calculus: Single-variable and multiple integrals
- Discrete Probability distributions and random variables (using calculus)
- Continuous Probability distributions and random variables (using calculus)

The lectures are designed to serve both as an introduction for participants who are new to these topics and also as a "review and reinforcement" refresher for those previously exposed to the material.

Proficiency in basic mathematics (e.g., rudimentary algebra and summation notation) is assumed but definitely not required. We will begin with a brief review of basic mathematics and statistics, which will include a question-and-answer session, to review and reinforce this material.

Since this is a Lecture workshop (as opposed to a Course), there are no grades or graded assignments. While there are no textbook or other supplemental readings, an overview of the material can be found (sometimes in an appendix) in many methodological textbooks. See me for more information about this.

I use Canvas to distribute this syllabus, optional "Worksheets" and corresponding "Tutorial and Answer Keys," and several other pdf documents. *I recommend that you also take notes during the lectures.*

The first Lecture workshop meeting is on Monday, July 07, 2025; it will be from 10:00 a.m. until 1:00 p.m. The remaining four Lecture workshops (Tuesday, July 08 - Friday, July 11) will be from 8:15 a.m. until 12:15 p.m. Of course, we will take a couple of breaks during these long lecture periods.

Time Zone

As I planned this Lecture workshop I was aware that we have participants in several time zones. Please note that all times on this syllabus, and all times referenced throughout this Lecture workshop, are stated in Ann Arbor (USA Eastern Daylight time zone [i.e., EDT; GMT/UTC -4h]) time.

This Lecture Workshop and Your Summer Program Courses

Being comfortable with the material in this Lecture workshop is important in many Summer Program courses. That is why you are attending these lectures! Therefore we will approach learning these mathematical tools from a perspective that is broad and generalizable, since they are applicable to several different (sometimes *very* different...) courses.

Given this broad scope of application areas, it is not optimal to provide narrow relevant examples that pertain to any specific course or discipline. So the answer to the (very reasonable...) question, “Can you show us where we would use this?” is generally (but not always!), “It is, or might be, discussed and used in some of the Summer Program courses.”

That said, to solidify some of these concepts I will sometimes provide examples and applications within a common, fundamental, and foundational baseline context: OLS linear regression.

So the material in this Lecture workshop is a means toward an end: Tools used to build a foundation for learning in some of the Courses you are (or will be) attending.

Using Canvas and Zoom

In addition to in-person meetings, we will use the University of Michigan Canvas platform for those choosing to attend this Lecture workshop online synchronously or asynchronously.

Documents (handouts, etc.) will be published on Canvas. I strongly recommend that you have access to these documents (and also take your own notes) during each Lecture workshop meeting session.

Synchronous Lectures can be accessed both in-person and also via Zoom on the University of Michigan Canvas platform. These lectures will also be recorded. The Teaching Assistants will monitor the Zoom Chat during each session.

Asynchronous Lecture Videos of each day’s lecture will be available in the “Media Gallery” on the University of Michigan Canvas platform as soon as possible after that day’s Lecture workshop ends.

Office Hours, for individuals or small groups, can be scheduled with me via communication on Canvas, email (mcdaniel@bvu.edu), or phone (+1 712 299 2587) and will be held either in-person or on Zoom.

Please remember that all times on this syllabus are stated in Ann Arbor, Michigan (i.e., USA Eastern Daylight Saving time zone [i.e., EDT; GMT/UTC -4h]) time.

This syllabus is continued on the next page

What is Covered?

This schedule is potentially subject to slight changes. So do not confuse "well organized" (which I am) with "inflexible" (which I definitely am not). For example, if we do not finish a topic on its assigned day then we will simply take our regular twenty-hour break and finish it at the start of the next day's class.

Here is the coverage schedule for the five days. *(For much more detail see the Outline on pages 3-7.)*

Monday, July 07 — Welcome to the Lecture Workshop...and Basic Stats...and Matrices

Tuesday, July 08 — Matrix Topics: Intermediate and Advanced

Wednesday, July 09 — Calculus: Differential (Derivatives) and Integrals

Thursday, July 10 — Probability: Continuous Random Variables

Friday, July 11 — Probability: Discrete Random Variables

Also: Remember that relevant documents for each Lecture workshop meeting are available in the Modules section of the University of Michigan Canvas platform. I *strongly* recommend that you have access to, and follow along with, the appropriate documents as you attend or view each lecture.

Concluding Remarks

Learning the material in this Lecture workshop will require a substantial amount of time, work, and energy on your part – but that is why you are here! I know the payoff will justify your effort.

It is an honor to be your instructor for this Lecture workshop.

Outline of Topics Covered

I. (Very) Basic Mathematics and Statistics: Brief Review and Question-and-Answer

II. Introduction to Matrices
- A. Definition
- B. Equality of Matrices
- C. Row Matrix and Column Matrix (Vector) (Array)
- D. Zero Matrix
- E. Transpose of a Matrix
- F. Square Matrices
  - 1. Definition
  - 2. Diagonal and Trace
  - 3. Symmetric Matrix
  - 4. Triangular Matrix
  - 5. Diagonal Matrix and Scalar Matrix
  - 6. Identity Matrix
- G. Partitioned (Block) Matrix
- H. Vocabulary Terms

III. Matrix Addition and Subtraction
  A. Conformability Conditions
  B. Matrix Addition
  C. Matrix Subtraction
  D. Properties of Matrix Addition and Subtraction
  E. Adding and Subtracting Partitioned Matrices

IV. Multiplication of a Scalar and a Matrix
  A. Notation
  B. Examples
  C. Matrix Negation
  D. Properties

V. Matrix Multiplication
  A. Terminology
    1. Left Matrix and Right Matrix
    2. Lead Matrix and Lag Matrix
    3. Pre-Multiplying and Post-Multiplying
  B. Conformability Conditions
    1. “Inner” and “Outer” Dimensions
    2. Conformability and Lack of Commutativity (i.e., “Order Matters”)
    3. Multiplication of a Matrix and its Transpose
    4. Multiplication Involving a 1x1 Matrix
  C. Method
    1. How to Do It: “Striped Shirt,” “Diving Board,” and “Point-and-Click” Explanations
    2. Examples of Matrix Multiplication Using Numbers
      a. General Examples
      b. Square Matrices: Lack of Commutativity (i.e., Order Still Matters)
      c. A Matrix and its Transpose (Noncommutative; Conformable, Square, Symmetric)
      d. Diagonal Matrices
      e. Multiplication by a Zero Matrix
  D. Substantive Examples
  E. More on the Identity Matrix
    1. Definition of a Multiplicative Identity: Scalar and Matrix
    2. Multiplication Involving the Identity Matrix
    3. Implicit Dimensions of the Identity Matrix (Whatever It Takes...)
  F. Powers of a Matrix
    1. Integer Powers
    2. Fractional Powers
  G. Idempotent Matrix
  H. Properties of Matrix Multiplication (Including Transpose Matrices)
  I. Systems of Linear Equations with Matrices
    1. Setup using Algebra
    2. The Coefficient Matrix, Variable Matrix, and Solution Matrix (CV=S)
  J. Online Matrix Calculator (matrixcalc.org/en/)

VI. Vectors
  A. Dot (Inner; Scalar) Product
  B. Magnitude (Norm)

VII. Kronecker (a.k.a. Tensor) Product: $A \otimes B$
  A. Definition
  B. Examples
  C. Properties

VIII. The Inverse of a Matrix
  A. Definition of a Multiplicative Inverse: Scalar and (Square) Matrix
  B. Notation for the Inverse Matrix
  C. Matrix Multiplication Involving an Inverse Matrix
  D. Systems of Linear Equations with Matrices Revisited: Solve It!
  E. Singular and Nonsingular Matrices
    1. Definitions
    2. Some Necessary Conditions for Nonsingularity
  F. Inverse of a Diagonal Matrix
  G. Properties of Inverse Matrices
  H. Calculating an Inverse Matrix: A Quick-and-Dirty Overview
    1. Gaussian Elimination Method
    2. Cofactor Method
  I. Example: The Inverse of the Correlation Matrix and Multicollinearity VIFs

IX. Determinant and Condition Number of a Matrix
  A. Determinant
    1. Existence and Notation
    2. Singularity and Nonsingularity
  B. Condition Number
  C. Calculating the Determinant: The 2×2 and 3×3 Cases

X. Rank of a Matrix
  A. Definition of Linear [In]Dependence
  B. Definition of Rank
  C. The Rank of a Square Matrix: Definition, Singularity, and Properties
  D. Multiplying a Matrix by Its Transpose: Implications Regarding (Non)Singularity

XI. Eigenvalues and Eigenvectors
  A. Definitions: Terms and Math
  B. Calculating: Examples
  C. Properties
  D. Ill-Conditioned Matrix

XII. Positive Definite Matrices
- A. Definitions: Terms and Math
- B. Calculation Examples
- C. Properties
- D. Singular Value Decomposition (SVD): A Very Brief Look

XIII. Variance-Covariance Matrices
- A. Definition and Types
- B. Properties and Uses

XIV. Using Matrices to Represent Linear Regression Models
- A. The Multiple OLS Regression Model
- B. Deriving the Estimation of the "B" Matrix
- C. The Standard Errors of the Slopes
- D. Rank, Singularity, and Multicollinearity
- E. Representing Data and Data Manipulations in Matrix Format

XV. Differential Calculus
- A. Explanation
- B. Involving a Constant
- C. Power Rule
- D. Sum and Difference Rule
- E. Exponentials and the Exponential Constant ("e")
- F. Natural Logarithms
- G. Product Rule
- H. Quotient Rule
- I. Chain Rule
- J. Partial Derivatives
- K. Second Derivative
- L. Concavity
- M. Minimums and Maximums

XVI. Integral Calculus
- A. Indefinite (Anti-Derivative)
- B. Power Rule
- C. Constant Rule
- D. Natural Logarithms and Exponentials
- E. Definite (Area)
- F. Limits of ∞ or $-\infty$
- G. Probability Density Function
- H. U-Substitution
- I. Integration by Parts
- J. Other Techniques
- K. Partial Integrals
- L. Multiple Integrals

XVII. Probability: Continuous Random Variables
- A. Introduction
- B. Expected Value
- C. Variance and Standard Deviation
- D. Integrals
- E. Uniform Distribution
- F. Normal Distribution
- G. Gamma Distribution
  - 1. The Mothership...
  - 2. The Gamma Function
  - 3. Why $0! = 1$
- H. Exponential Distribution
- I. Chi-Square Distribution
- J. Beta Distribution

XVIII. Moment Generating Functions
- A. Definition
- B. Calculations
  - 1. Mean
  - 2. Variance and Standard Deviation
- C. Examples

XIX. Probability: Discrete Random Variables
- A. Introduction
- B. Expected Value
- C. Variance and Standard Deviation
- D. Counting (“a Choose b”)
- E. Summations
- F. Binomial (and Bernoulli) Distribution
- G. Poisson Distribution
- H. Geometric Distribution
- I. Negative Binomial Distribution
- J. Hypergeometric Distribution

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